

Finite Models

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Having finished our discussion on classical logic on arbitrary models, we now turn specifically to finite models. This is a whole subarea of logic, called Finite Model Theory.

Motivation: Many situations, especially in CS, use only finite models. For example, finite graphs, databases, and so on.

What is different moving to finite models from classical logic? *Basically everything!*

- Validity
- Satisfiability
- Implication
- ... All different!

1 Example

Here is a set of sentences that is satisfiable on arbitrary models but not on finite models: $\Sigma = \{\sigma_k \mid k > 0\}$, where $\sigma_k = \exists x_1 \dots \exists x_k (\bigwedge_{i \neq j} x_i \neq x_j)$

In some sense, this is “cheating” by using an infinite set; is there actually a single sentence with this property?

Yes, there is. Here is a sentence which is satisfiable on infinite models but not finite ones:

Vocabulary: Binary relation R

Intuitively, the sentence says that R is a dense total order.

$\sigma : \exists x \exists y R(x, y) \wedge$ $\forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z)) \wedge$ $\forall x \forall y (x = y \vee \neg(R(x, y) \wedge R(y, x))) \wedge$ $\forall x \forall y (R(x, y) \vee R(y, x))$ $\forall x \forall y \exists z (R(x, y) \rightarrow (R(x, z) \wedge R(z, y)))$	<div style="display: flex; align-items: center;"> <div style="border-left: 1px solid black; border-right: 1px solid black; padding: 0 5px; margin-right: 5px;">The relation and set are non-empty</div> <div style="border-left: 1px solid black; padding: 0 5px; margin-right: 5px;">R is transitive</div> <div style="border-left: 1px solid black; padding: 0 5px; margin-right: 5px;">R is antisymmetric</div> <div style="border-left: 1px solid black; padding: 0 5px; margin-right: 5px;">R is total</div> <div style="border-left: 1px solid black; padding: 0 5px;">R is dense</div> </div>
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Claim: σ has an infinite model: namely, \mathbf{Q} .

On the other hand, σ has no finite model M . In fact, we can show by induction that for every n , if model M satisfying σ contains n elements, say $a_1 < a_2, \dots < a_n$ (with $<$ giving the relation imposed by R), it must contain yet another element, e.g. an element $b, a_1 < b < a_2$. Thus, M must be infinite.

2 Gödel's Completeness Theorem in the Finite Case?

Now we consider this question: is there an analogue of Gödel's completeness theorem for validity in the finite case?

Recall: $\text{VALID} = \{ \text{sentences that are valid on all models} \}$.

Let VALID_{fin} represent finitely valid sentences; those that are true on all finite models.

Let SAT_{fin} be sentences that have some finite model.

Is VALID_{fin} r.e.? No.

We will show later that finite satisfiability is undecidable by showing SAT_{fin} is not recursive.

From this, it follows immediately that VALID_{fin} is not r.e.

Proof by contradiction; suppose VALID_{fin} is r.e.

Then UNSAT_{fin} is also r.e. Why? Note that a sentence is unsatisfiable if its negation is valid. Let us enumerate all valid formulas; whenever we produce one of the syntactic form $\neg\varphi$ output the formula φ , which is by definition unsatisfiable. Since all valid FO sentences have proofs, this process enumerates all unsatisfiable sentences.

On the other hand, it is true that SAT_{fin} is r.e.: To see this, let us enumerate all sentences and finite models in parallel. Whenever we find φ that satisfies one of the finite models so far enumerated, output φ . If φ has a finite model, this will be eventually found, and φ will be output.

Together, these two results imply that SAT_{fin} is recursive.

However, SAT_{fin} is not recursive, a contradiction. Thus VALID_{fin} is not r.e.

Compactness also fails in the finite case. Let $\Sigma = \{\sigma_k | k \geq 1\}$, where σ_k states that the size of the universe is $\geq k$. Every finite $\Sigma_o \subseteq \Sigma$ has a finite model. However, Σ obviously has no finite model.

Compactness is an important tool, used all the time in classical logic. Without it, we cannot, for example, use compactness to show that connectivity of finite graphs is not first-order definable (as we did before).

We now return to show that finite satisfiability is undecidable (not recursive) Note: the proof is in Libkin's Finite Model Theory book, as well as the Foundations of Databases book.