Assignment 3-4

1. (4 points) Consider a vocabulary consisting of a single binary function $*$ and the constant 1. The following three sentences over this vocabulary are called the axioms of group theory (we use infix notation for $*$):

- $\forall x \forall y \forall z ((x * y) * z = x * (y * z))$ (associativity)
- $\forall x (x * 1 = x)$ (1 is the identity)
- $\forall x \exists y \ (x * y = 1)$ (existence of inverses).

A group is any structure over vocabulary $*$, 1, satisfying the above sentences. The three axioms form a complete axiomatization of groups: every sentence satisfied by all groups is provable from the axioms. It is also known that the theory of groups is undecidable (i.e. it is undecidable if a sentence is true for all groups). Now compare this with the situation for the integers. When we proved Gödel's Incompleteness Theorem we showed that if there was a complete axiomatization for the integers then it would be decidable if a sentence is true for the integers. Why doesn't our proof work for the theory of groups? Explain what is different.

2. (5 points) Recall that an FO sentence has the *finite model property* if either it has no model or it has at least one finite model. Show that it is undecidable if an FO sentence has the finite model property.

3. (5 points) The movie database consists of the following two relations

movie: title, director, actor
schedule: theater, title

The first relation provides titles, directors, and actors of various movies. Assume a movie is uniquely identified by its title. The second relation provides the titles of currently playing movies and the theaters where they are being shown. Express the following queries in FO and relational algebra:

(a) (2 points) List the theaters showing some movie by Hitchcock.

(b) (3 points) List the theaters showing only movies by Hitchcock.

4. (5 points) Use Ehrenfeucht-Fraissé games to show that there is no FO sentence over vocabulary $(<, =)$ that distinguishes[1] the real numbers (with ordering on the reals) from the rational numbers (with the ordering on the rationals). Now suppose the vocabulary is extended to include $+$. Explain informally why the problem is harder in this case.

$\star$ **5.** (5 points) Let E be a binary relation (providing edges in a graph) and let L_n denote an interpretation of E consisting of a simple path of n elements. Consider the sets of integers of the form $N_\varphi = \{n \mid L_n \models \varphi\}$, where φ is an FO sentence over vocabulary E. Describe the sets N_φ. (In other words, characterize the sets of integers that equal N_φ for some φ.)

6. (6 points) Determine whether the following properties of directed graphs are almost surely true or whether they are almost surely false.

(a) Existence of a cycle of length three.

(b) Connectivity.

(c) Being a tree.

Hint: Try to avoid brute-force counting; use what you already know.

7. (5 points) This provides an example showing that FO no longer has a 0-1 law if functions are allowed. Consider the vocabulary consisting of one unary function f. Let σ be the sentence $\forall x(\neg(x = f(x))$. Prove that σ has asymptotic probability $\frac{1}{e}$.

[1] Meaning that it is true for one but not the other.