

CSE 260 - Introduction to Parallel Computation

Class 4 - October 2, 2001

PDE's for Dummies

Disclaimer!

This methods and opinions expressed in this lecture are solely mine, and do not reflect the collective wisdom of mathematicians, computational scientists, or my great grandfather (who wrote an algebra textbook).

Nevertheless, you're going to be held responsible for them in the next quizlet!

Four Steps of Computational Science

1. Model some physical phenomenon by partitioning it into tiny cells and considering forces over tiny timesteps.
2. Let "tiny" go to zero, use techniques of calculus to find differential equations.
3. Solve these differential equations by reintroducing tiny cells and simulating forces over tiny timesteps.
4. See if results appear to be correct.

Dummy's three-step method

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What is the value of step 2?

- It's easier for scientists who know what they are doing to formulate a model using partial differential equations.
- Mathematicians have proven theorems about how big "tiny" can be and still lead to valid answers.
- Allows a different method in step 3 from step 1.
- The scientists get to write compact equations using esoteric symbols.

$\delta \quad \Delta \quad \Psi \quad \Sigma \quad \varphi \quad \Gamma \quad \chi \quad \rho \quad \oplus \quad \nabla$

What's the disadvantage of step 2

- It may hide the assumptions used in making the model.
- The needed theorems may not exist or not be known.
- Most computer scientists are intimidated by compact equations using esoteric symbols.

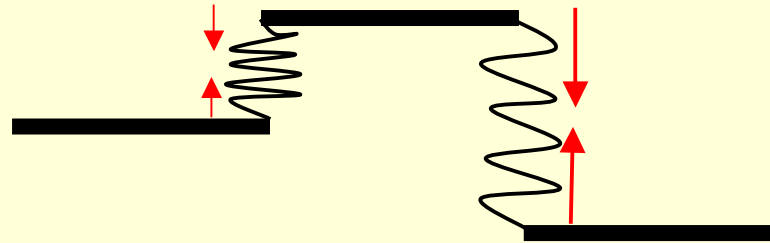
$\delta \quad \Delta \quad \Psi \quad \Sigma \quad \varphi \quad \Gamma \quad \chi \quad \rho \quad \oplus \quad \nabla$

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Example: Simulating Waves

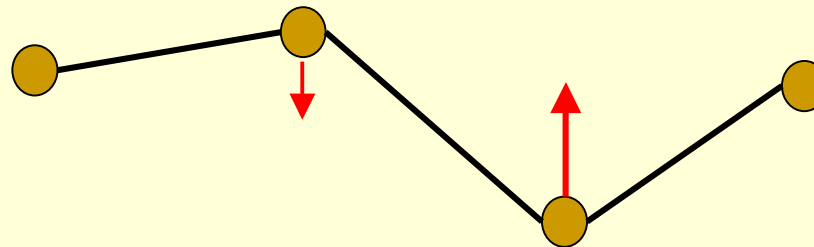
Model #1:

wire = horizontal line segments & vertical springs



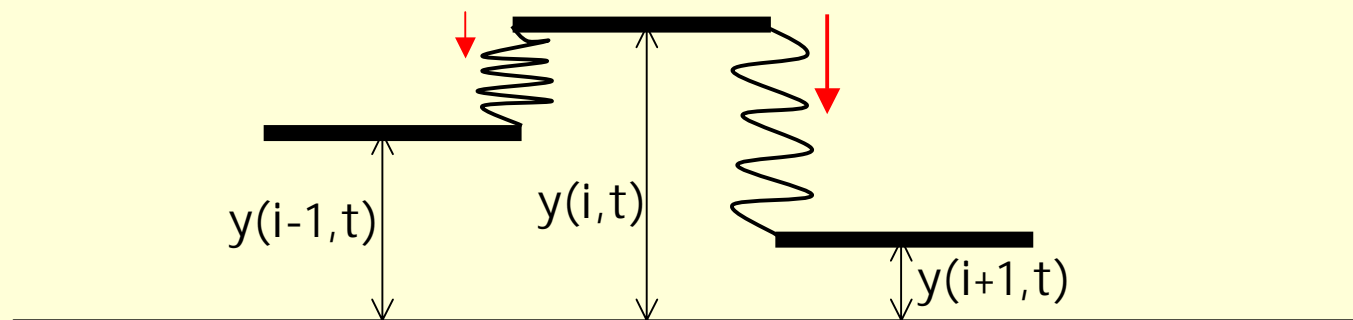
Model #2:

wire = string of beads; only look at vertical forces



Model 1 forces

Denote height of i^{th} segment at timestep t by $y(i,t)$



Assume spring exerts force proportional to distance stretched.

$$\begin{aligned}\text{Upward force on } i^{\text{th}} \text{ segment} &= c(y(i-1,t) - y(i,t)) + c(y(i+1,t) - y(i,t)) \\ &= c(y(i-1,t) - 2y(i,t) + y(i+1,t))\end{aligned}$$

constant (depends on material and length of segment)

Model 1 motion (in Δt timestep)

Newton: An object in motion remains in motion ...

$$y(i,t+1) = y(i,t) + \underbrace{(y(i,t) - y(i,t-1))}_{\text{distance traveled in previous timestep}}$$

distance traveled in previous timestep

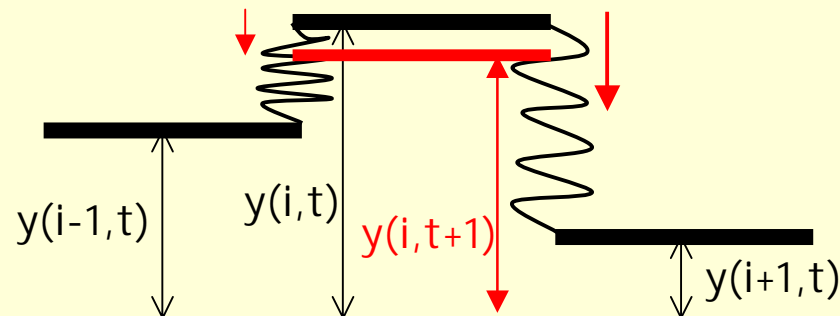
....unless acted upon by an outside force.

$$f = ma \text{ and } d = \frac{1}{2} at^2 \text{ yield } d = \frac{1}{2} f(\Delta t)^2/m$$

mass of a segment

Putting it all together gives:

$$y(i,t+1) = 2y(i,t) - y(i,t-1) + (c/2m)(\Delta t)^2(y(i-1,t) - 2y(i,t) + y(i+1,t))$$



The dummies are done!

This formula:

$$y(i,t+1) = 2y(i,t) - y(i,t-1) + (c/2m)(\Delta t)^2(y(i-1,t) - 2y(i,t) + y(i+1,t))$$

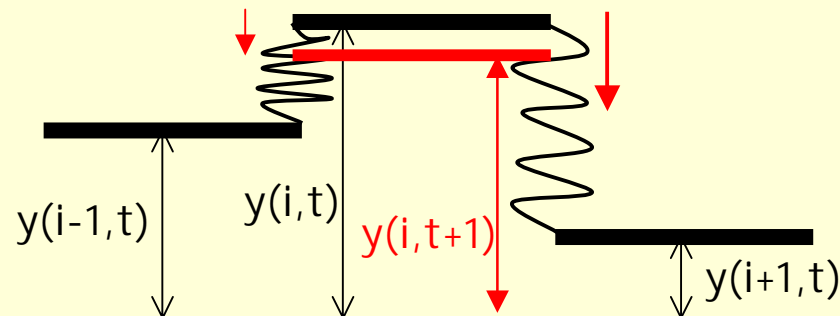
lets us compute all the $y(\cdot, t+1)$'s given the previous values.

Note that we can rewrite it as (for suitable constants c_j):

$$y(i,t+1) = c_1 y(i,t) + c_2 y(i,t-1) + c_3 y(i-1,t) + c_3 y(i+1,t)$$

Given initial values for $y(\cdot, 0)$, we can simulate forward in time.

(We'll need to add "boundary conditions" on $y(0, \cdot)$ and $y(N, \cdot)$.)



What does calculus say

Rewrite:

$$y(i,t+1) = 2y(i,t) - y(i,t-1) + (c/2m)(\Delta t)^2(y(i-1,t) - 2y(i,t) + y(i+1,t))$$

using $c = s/\Delta x$ and $m = d\Delta x$ and $u(x,t) = y(x/\Delta x, t/\Delta t)$

stiffness of wire

density of wire

to get:

$$(y(i,t+1) - 2y(i,t) + y(i,t-1))/(\Delta t)^2 = (s/2d) (y(i-1,t) - 2y(i,t) + y(i+1,t))/\Delta x^2$$

approximation to second derivative of u w.r.t t

approximation to second derivative w.r.t x

or, using funny symbols, $\frac{\delta^2 u}{\delta t^2}(x,t) = (s/2d) \frac{\delta^2 u}{\delta x^2}(x,t)$

What about step 4??

“See if results appear to be correct.”

Compare to experiments.

Compare to simulations using a better model.

Why not use the better model in the first place??

How do you compare different answers?

Answers differ after the slightest change!

Examine properties that don't change much

e.g., frequencies of vibration.

Or see if pictures or movies look right.

Model #2

(On board. PowerPoint equations are tedious!)

What does the 4-step method say*?

"Substituting these formulae give a horrendous mess. However, we can get considerable simplification by looking at small vibrations."

So they assume $\sin \Theta(x,t) \approx (y(x,t) - y(x-1,t)) / \Delta x$

In other words, they switch to model 1 in the middle.

* From www.math.ubc.ca/~feldman/apps/wave.pdf

Model #3

An more realistic model would allow the beads on the string to move horizontally as well as vertically.

Dummy method could handle this (good project!) by having the inner loop doing trigonometry (slow, but accurate).

The “traditional” method says (near bottom of page 2),

“As a second simplification, we assume that ... our tiny string element moves only vertically.”

Precariously, it continues

“Then the net horizontal force on it must be zero.”

(This seems flaky – it’s one thing to ignore an effect, hoping it is small, but to make a false assumption and then use it to gain another equation seems very dangerous!)

More detail: www-ccrma.stanford.edu/~jos/waveguide/More_Complete.Derivation.html

Recommended exercise:

Repeat the Dummy method for a 2-D sheet.

Note: next week's quizlet will be to do a different (fairly simple) Dummy-style derivation!

I don't require you do the "calculus" part, but if you can, then please do so!