The Reality of Mathematical Objects

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Abstract

Many mathematicians & philosophers say math objects have real existence independent of representation, & of all human society.
But do mathematicians really behave as if this were true?

Use cognitive linguistics, discourse analysis, ethnomethodology, semiotics:

— How do math objects enter math discourse?
— How are math objects referred to in math discourse?
— What does this tell us?

We will see that math discourse is:
highly nuanced for object status,
highly structured, and that
math objects have attached values.

There are implications for:

— philosophy of math,
— math education, exposition & use.

We can also infer some values of mathematics.

Outline

1. Preface
2. Modes of Object Introduction
3. Modes of Object Reference
4. Sequential Organization
5. Cognitive Linguistics of Math
6. Some Literature
7. Conclusions

1. Preface

Two paradoxes of mathematics:

1. Real math is embodied, situated, material,
   but math objects appear to be objective & transcendental.
2. Although very abstract, math is very applicable (Wigner).

We will see:

- how transcendental objectivity is achieved by discourse practice;
- that math is grounded in experience with everyday world;
- that math discourse expresses the values of provers.

We will see how these & similar points resolve the paradoxes.

Based on empirical study of "math objects" in "real math,"
in sense of actual practice of professional mathematics,
using written material & live videos,
examining structure in math discourse,
drawing on member's competence in math.

This is a case study of social construction of objects & how values get attached to them by how they are used.

We will examine

- how "objects" are introduced,
- how they are referenced,
- narrative structure of math discourse,
- what these tell about math objects.

Draw on cognitive linguistics, discourse analysis, semiotics, and ethnomethodology, but non-dogmatically,
use empirically well grounded relevant theories.

Proofs are locally adequate for given practical purpose,
e.g., only elaborated to extent needed for that occasion.

Proving is accountable in sense of making clear what it is;
accountability is a natural social achievement (for members).

Look for deviations from expected structure and ask:
What work is this doing?

Values are attached by highly nuanced modes of reference, including

- degree of existence,
- importance,
- difficulty,

as determined by syntax, discourse, sequential organization
— can even be rhetorical & dramatic effects.

(“Scope” and “role” of objects are technical,
related to tree structure of open goals, and to methods of proving.)

Math symbols are tokens in math discourse,
introduced for future use.

To say they refer to “real” objects is nonsense —
we have no access to Platonic heaven,
but doing math feels as if we do.
Why is that?

This work fits into larger project on “natural ethics”:
recovering deeply embedded values from natural interaction,
by asking what work is being done.

2. Modes of Introduction

Let N be an integer.	
Given an integer N, ...	
Suppose N is an integer, ...	*Assume that N is an integer.*
Assuming N is an integer, ...	*Granting that N is an integer, ...*
For N an integer, ...	
If N is an integer, ...	
..., where N is an integer.	*..., with N an integer.*
..., assuming N is an integer.	
..., provided N is an integer.	

Syntax includes main vs. subordinate clause, fronting:

- main clause & fronting strengthen an introduction;
- deeper subordination weakens it.

Delicate shadings of discourse salience, including
values of importance, difficulty & “ontological status;”
latter ranges from creation & baptism to possible non-existence.

Main activity of professional math is exploration, not exposition
— in exploration, often don't know if objects really exist
— "suppose," "assume," "granting," etc. reflect this uncertainty.

Also
"thus we see"
"and so we find"
"thus we discover"
etc.
reflect a metaphor of exploration, not of creation.
(But creation metaphor can occur in constructions, e.g.,
"We drop a perpindicular."

Rhetoric of truth & reality used when proof succeeds, i.e.,
truth is mathematicians word for what happens in
properly accountable proving.

Signifier & signified may be identified or not:

..., where N denotes an integer. *..., where N indicates an integer.*

..., where N ranges over integers.

Objects need not be introduced before used; instead can follow general conventions:

We let capital letters denote sets.

We will use the notation $\vec{a}, \vec{b}, \ldots$ for vectors.

Objects not always given symbol when introduced:

We will operate over a fixed real closed field.

We work in an arbitrary but fixed Hilbert space.

Objects may be introduced with assumptions:

Let N be a positive integer.

..., where P is a prime greater than N.

Results & assertions are also introduced:

We claim that …
Then, therefore, thus, hence, …
Observe that … *Notice that …* *Note that …*
We will prove that …
It follows that … *It can be shown that …*
It is clear that … *It is obvious that …*
It is easy to see that … *It is easy to prove that …*
It is easily seen that …
A [little] calculation shows that …
The reader can [easily] check that …
It is possible to prove that …
Theorem, **Proposition**, **Lemma**, …

Delicate shadings of importance & difficulty.

"*It is easy to see*" often indicates a tricky or tedious calculation
— this is well known to professionals, but irritating to others.

Proofs by contradiction show a thing assumed to exist really does not; can be very confusing to newbies.

We will show that there are no even primes greater than 2.

Suppose p is an even prime greater than 2.

Then $p = 2n$ for some $n > 1$.

Therefore p is not prime.

Here is a more "dramatic" version:

I claim there are no even primes greater than 2.

Let p be an even prime greater than 2.

Then $p = 2n$ for some $n > 1$.

Therefore p is not prime.

QED

Proofs based on logic differ from those based on concrete constructions; apply to all objects that satisfy the assumptions (if any);

e.g., constructions of Euclidean geometry vs. group theory.

3. Modes of Reference

Math discourse builds on ordinary discourse
– shares its conventions, including modes of reference.

Ordinary discourse presupposes concrete objects, referred to by:
"it," "that," "this," "the other," etc.

In math, also have
"this formula," "that integer," "the other variable," etc.

Can also have chains of reference.
"*The formula that we used to prove this one.*"

Default is do what takes least effort & still effective;
differences from this do some other work in discourse,
such as importance, difficulty, "ontological status".

Meaning of reference controlled by importance, difficulty, scope:
"The proof is now reduced to ..."
"The desired result now follows."

Reference can be rather complex in multimedia discourse:

"The last formula with x replaced by y."

"The last formula with <u>this</u> and <u>that</u> reversed."

"The last formula with <u>this</u> substituted for <u>x</u>."

"The formula that used to be here."

where underlined parts are gestures, writing, writing plus gesture, or ....
Importance, etc. indicated by body language in complex ways...

Similar phenomena for introducing formulae:

"Consider the formula <u>...</u>" *"Therefore <u>...</u>"*

"What we want to prove is <u>...</u>" *"And so we are reduced to <u>...</u>"*

Speaking & pointing can be simultaneous in these.

Of course, can also have named formulae:

"Let F be the following formula" *"Let F be the above formula."*

"Let F denote the following formula" *"Let F be the formula <u>...</u>"*

Named formulae more common in writing than live discourse. Also

$$[6.21] \qquad e^{\pi i} = -1 \; .$$

Math objects are abstract in the sense that
there are multiple representations, considered equivalent.
For example, the following

$$\begin{aligned} \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} &= \\ \tfrac{6}{12} + \tfrac{4}{12} + \tfrac{3}{12} &= \\ \tfrac{13}{12} &= 1\tfrac{1}{12} \end{aligned}$$

are all “equal” — but are still all different.

The work of calculation often consists of changing representation.
The calculator tends to think such terms represent “the same thing,”
— which reinforces belief in a Platonic object.

Use of equivalence is ubiquitious in math:

$$\begin{aligned} .99999... &= 1 = 1.0 \\ .33333... &= \tfrac{1}{3} \\ 0 + x &= x \\ 0 \times x &= 0 \end{aligned}$$

4. Sequential Organization

Ideas from Conversational Analysis (part of ethmeth) can help, since have sequential organization of introduction, reference & re-reference:

- reference can be simpler each time, as salience rises.
- repair conventions also used for real-time editing of formulae.

According to Labov, stories have:

1. an optional orientation;
2. narrative clauses for actions,
3. in narrative past tense,
4. with events presumed in “narrative order”;
5. evaluative clauses for reasons; and
6. an optional closing section.

(Later refined by Linde.)

Narrative structure often appears in proofs,
used with Logical Consequence is Temporal Succession metaphor:
"then", "it follows that", "after which", ...

I claim more narrative structure can make proofs easier to follow:
tested in proof display system on the web.

A dramatic example on video tape:
a heroic narrative of proof attempts by computer program.
a bit like Joseph Campbell's ideas.

Narrative structure also adds to illusion of reality of math objects,
again using conventional discourse structures,
that ordinarily involve concrete objects & events.

5. Cognitive Linguistics of Math

Brilliant new book by Lakoff & Núñez —
shows math objects grounded in everyday experience:

- basic image schemas (based on sensory-motor schemas);
- metaphoric projection;
- conceptual blending.

Also, cognitive processing of language is largely unconscious.

Examples for numbers:

"add," "take away," "yields," "get," "bigger than," ...
— based on Numbers Are Object Collections metaphor.

"x is between 5 and 6," "x is below 12," "y is close to x," "x is far from 0," ...
— based on Numbers Are Points on a Line metaphor.

These also reinforce illusion of reality of math objects.

Basic image schemas explain applicability of math:

sensory-motor schemas work in the world (due to evolution);
so does language abstracted from them;
math is just a bit more abstracted.

Wigner's paradox depends on accepting that math is transcendental.

"*Let N be an integer*" calls up a "conceptual space":

– more than just concepts;
– relations among concepts, such as $\leq$;
– local methods, e.g., practices for proving;
– may even be an evolving local state, e.g., geometrical constructions.

Similar to object oriented programming in computer science.
(This goes beyond Lakoff & Núñez).

Lakoff & Núñez also use conceptual metaphors & blends;
this too suggests reality of math objects.

Their book has many good examples.

6. Some Literature

Literature a bit chaotic; much is speculative, normative, or irrelevant; not much based on empirical studies of actual situated mathematics.

- Famous mathematicians pontificating,
  based on experience with very hard proofs.
  Like advice from Olympic athletes – not useful to ordinary folks – could well be harmful – e.g., smoking for algebra!
- Philosophers, based on (often ridiculous) ideas of what is real.
- Educationists, based on classroom experience, often strange mixtures of theories, & sometimes hidden agendas.
- Psychologists, based on artificial lab experiments.

6.1 Lakoff & Núñez

Focussed on embodied aspects of math.
Tool kit includes image schemas, conceptual metaphors, blends.
No analysis of discourse as such; no attention to multimedia.
No sequential analysis, e.g., introduction & reference, narrative.
Good on historical development of concepts & metaphorical projection
Excellent analysis of math concepts – not part of present study.
Perhaps a bit polemical at times.

6.2 Livingston

Great older book on ethnomethodology of mathematics.
May be frustrating, since stays in framework of working math,
doesn't reach what most would call conclusions.
Has an excellent example: Gödel's incompleteness theorem.

6.3 Sfard

Focused on education, especially on learning new concepts.
Also considers historical development of math concepts.
Just one case study, but an interesting one.
Anticipates several key points from Lakoff & Núñez.
Uses an odd mixture of theories, including older Lakoff work.

6.4 Goguen

Paper *Towards a Social, Ethical Theory of Information*:
Foundations for notion of information based on social semiotics, with examples from computer systems design.
Uses ideas from ethnomethodology & sociology of science, argues that ethics are inherent in interaction.

Also *The Ethics of Databases*: on values in user interfaces, with examples from popular web search engines.

7. Conclusions

Math discourse is finely nuanced for salience,
in both introduction & reference,
reflecting values of existence, importance & difficulty.

Math objects constituted & sustained as real by practical work in actual discourse, by competent members.

The reality of mathematical objects is an illusion.

Due to conventions of ordinary discourse,
which presume ordinary concrete objects,
including sequential organization of object introduction & reference,
narrative, image schemas, metaphors, blends, abstraction & more.

Most mathematicians know reality is an illusion
— but are trained not to admit it
— because believing the illusion helps them do math better!

1. Platonism (also called “realism”) is false:
   - real math is embodied & situated;
   - Platonic objects have no causal role in real math discourse.
2. Formalism is false:
   - real math discourse is not only informal,
     is highly nuanced, expressive, even dramatic;
   - real math is grounded in real experience;
   - traces exist in language of math, e.g., image schemas.
3. Real math is not routine:
   - proofs are site specific,
     e.g., only elaborated to extent needed at that time & place.
4. Some implications for math education:
   - “Back to basics” – memorization, routinization not good teaching.
   - Grounding everything in practical experience is misguided.
   - Teaching math as pure abstraction is misguided.

Similar points for exposition & use of math.

Also for physics, chemistry, engineering, e.g., concepts like “force.”

Some really new results here, including:

- Use of difficulty & existence in resolving references.
- Metaphoric projection of narrative presumption to entailment.
- Large range of linguistic devices in the rhetoric of reality.
- The role of values.

In a way, we used the work done to make values invisible
to render them visible.
(A little too cute – but some insight there.)

Can also go up a level & infer some values of provers:

1. clarity of reference
2. clarity of organization
3. difficulty of proof
4. concision
5. austerity
6. surprise.

(But no room for supporting aguments here.)