Quiz 1 Solutions

Problem 1 [16 points] Let

$$A = \left\{ w \in \{0,1\}^* : \begin{array}{l} \text{Either } w \text{ begins with a 0 and contains at least one 1,} \\ \text{or } w \text{ ends with a 1 and contains at least one 0} \end{array} \right\}$$

In the box below, write a regular expression describing the language A:

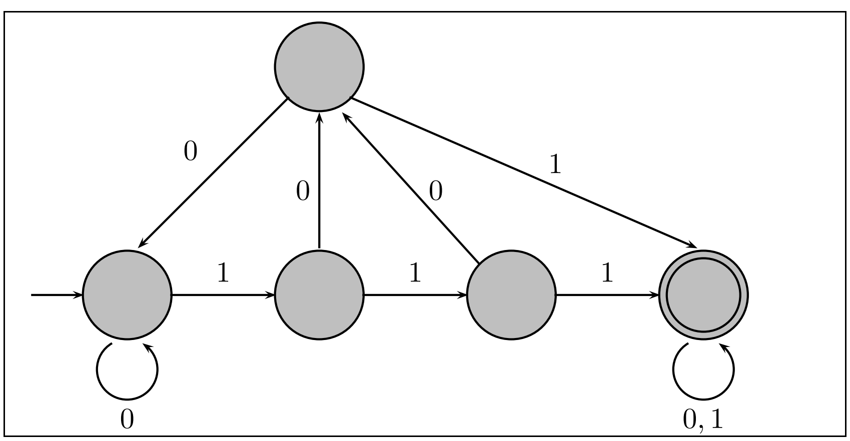
$$0(0 \cup 1)^*1(0 \cup 1)^* \ \cup \ (0 \cup 1)^*0(0 \cup 1)^*1$$

In the first case, w is a string beginning with a 0, followed by any string consisting of zero or more ones and zeros, followed by a 1, followed again by any string consisting of zero or more ones and zeros. In the second case, w is a string consisting of any string of zero or more ones and zeros, followed by a 0, followed by any string of zero or more ones and zeros, followed by a 1.

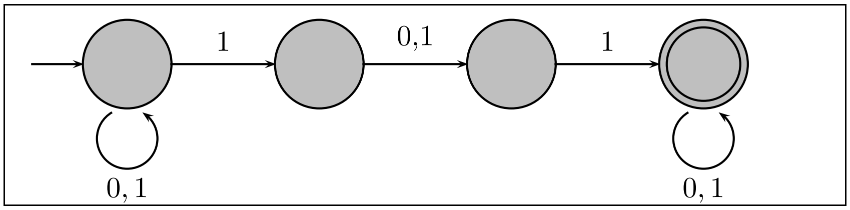
Problem 2 [30 points] Let

$$A = \{ w \in \{0,1\}^* : w \text{ contains either 111 or 101 as a substring} \} .$$

1. **[15 points]** Draw the state diagram of a DFA with *at most five states* that recognizes A.



2. **[15 points]** Draw the state diagram of a NFA with *at most four states* that recognizes A.



Problem 3 [32 points] If $w \in \{0,1\}^*$ is a string then $\mathsf{flip}(w)$ is the string obtained by flipping each bit of w. (For example, $\mathsf{flip}(01101) = 10010$). If $A \subseteq \{0,1\}^*$ is a language, we let

$$\begin{aligned} B &= \{ w \in \{0,1\}^* : \mathsf{flip}(w) \in A \} \\ C &= \{ w \in A : \mathsf{flip}(w) \notin A \} . \end{aligned}$$

Assuming A is regular, prove the following:

1. **[16 points]** B is regular.

We use the template for proofs of closure properties that we have used many times in class.

Given: A is regular. So there is a DFA $M = (Q, \{0,1\}, \delta, q_0, F)$ that accepts A.

Want: To show that B is regular. We will do this by constructing a DFA N that recognizes B. (It would suffice to construct an NFA, but in this case it is just as easy to construct a DFA, so we do.)

Construction: We simply flip the labels on the arrows of M, turning ones into zeros and zeros into ones. Formally, our DFA is $N = (Q, \{0,1\}, \delta', q_0, F)$, meaning all components are the same as in M except for the transition function. The new transition function is defined for all $q \in Q$ and all $\sigma \in \{0,1\}$ by

$$\delta'(q,\sigma) = \begin{cases} \delta(q,0) & \text{if } \sigma = 1 \\ \delta(q,1) & \text{if } \sigma = 0 . \end{cases}$$

Correctness of construction: M goes from q_0 to a state f on an input w if and only if N goes from q_0 to f on input $\mathsf{flip}(w)$. So M accepts w if and only if N accepts $\mathsf{flip}(w)$.

2. **[16 points]** C is regular. (*Hint:* Use the fact that B is regular. You may do so even if you did not prove this.)

A string w is in C exactly when the following two conditions are both met: (1) w is in A, and (2) $\mathsf{flip}(w) \notin A$, meaning w is not in B. This means that $C = A \cap \overline{B}$. Now, we can show that C is regular by using known closure properties of the class of regular languages:

Claim	**Justification**
$\overline{B}$ is regular	B is regular, and we know that L regular implies $\overline{L}$ regular for any language L
A is regular	by assumption
$A \cap \overline{B}$ is regular	$A, \overline{B}$ are both regular, and we know that L_1, L_2 regular implies $L_1 \cap L_2$ regular for any languages L_1, L_2
C is regular	$C = A \cap \overline{B}$

Problem 4 [22 points] Recall that $|x|$ denotes the length of a string x, and let

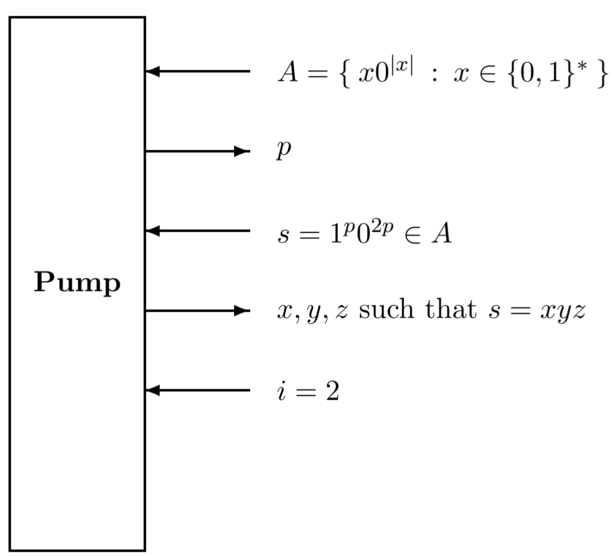
$$A \;=\; \{\, x0^{2|x|} \;:\; x \in \{0,1\}^* \,\} \,.$$

Prove that A is not regular.

We use the usual template. The proof is by contradiction.

Assume: $A = \{\, x0^{2|x|} \;:\; x \in \{0,1\}^* \,\}$ is regular.

The assumption means that the pumping lemma (Theorem 1.37, page 78 of the text) applies to A. We imagine ourselves "interacting" with the lemma as follows:



We give it A, and it returns a pumping length p. Now, we choose a string $s \in A$ of length greater than p, and return it to the lemma. The choice of string is important for the rest of the argument, and we set it to $s = 1^p0^{2p}$. This is in A, because $w = 1^p$ is a string of length p, and thus s is $w0^{2|w|}$. Because s has a length greater than p, the pumping lemma says that s can be split into xyz which obey the three conditions of the pumping lemma. The lemma returns x, y, z to us. We then choose $i = 2$ and return it to the lemma. At this point, the lemma guarantees that

(1) $xy^2z \in A$

(2) $|y| > 0$

(3) $|xy| \leq p$

Because the first p symbols of s are all 1, we know by condition (3) above that x and y must contain only ones. Condition (2) states that y must have length greater than zero, so we know y contains at least one one. So there exist a, b, c such that $x = 1^a$, $y = 1^b$ and $z = 1^{p-a-b}0^{2p}$ and $b \geq 1$ and $a + b \leq p$. So

$$xy^2z = 1^a1^b1^b1^{p-a-b}0^{2p} = 1^{a+2b+p-a-b}0^{2p} = 1^{p+b}0^{2p} .$$

However, there is no string w such that $1^{p+b}0^{2p} = w0^{2|w|}$, because the only choice for w would be 1^{p+b} which has length $p+b \geq p+1$. So the definition of A tells us that $xy^2z \notin A$. But condition (1) says $xy^2z \in A$. They can't both be true, so we have a contradiction. This means our assumption that A was regular is false.
