

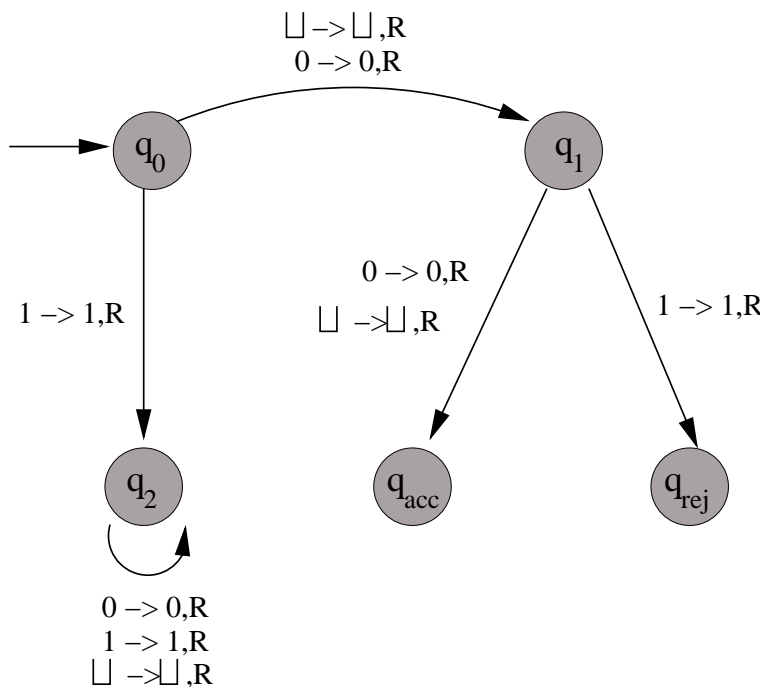
## Quiz 3 Solutions

Figures 1 and 2, at the end of this document, reproduce the tables you were provided on Quiz 3 and could use in your solutions.

**Problem 1 [20 points]** Draw the state diagram of a TM  $M$  with input alphabet  $\{0, 1\}$  and the following behavior on any input  $w \in \{0, 1\}^*$ :

If $w$ begins with	then $M(w)$
00	accepts
01	rejects
10	loops
11	loops

Your TM should have *at most 6 states*.



**Problem 2 [20 points]** Recall that  $|w|$  denotes the length of a string  $w$ . Prove that if  $A$  is decidable then so is

$$B = \{ w \in \{0,1\}^* : \exists x \in \{0,1\}^* \text{ such that } |x| = |w| \text{ and } x \in A \}.$$

We use the usual template for proofs of closure properties.

**Given:** Assume  $A$  is decidable. This means that there is a TM  $M$  which decides  $A$ . This in turn means that for all strings  $x \in \Sigma^*$

- If  $x \in A$  then  $M(x)$  accepts.
- If  $x \notin A$  then  $M(x)$  rejects.

**Want:** To show that  $B$  is decidable. To do this we need a TM  $N$  that decides  $B$ , meaning for all  $w \in \Sigma^*$

- If  $\exists x \in \{0,1\}^*$  such that  $|x| = |w|$  and  $x \in A$ , then  $N(w)$  accepts.
- If there is no  $x \in \{0,1\}^*$  such that  $|x| = |w|$  and  $x \in A$ , then  $N(w)$  rejects.

**Construction:**  $N =$  On input a string  $w$

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Let  $n = |w|$  be the length of  $w$ 
For all  $n$ -bit binary strings  $x$  do
    If  $M(x)$  accepts then accept
EndFor
Reject

```

**Problem 3 [20 points]** Recall that  $|w|$  denotes the length of a string  $w$ . Prove that if  $A$  is recognizable then so is

$$B = \{ w \in \{0,1\}^* : \text{Either } 0^{|w|} \in A \text{ or } 1^{|w|} \in A \}.$$

We use the usual template for proofs of closure properties.

**Given:** Assume  $A$  is recognizable. This means that there is a TM  $M$  which recognizes  $A$ . This in turn means that for all strings  $x \in \Sigma^*$

- If  $x \in A$  then  $M(x)$  accepts.
- If  $x \notin A$  then  $M(x)$  rejects or loops.

**Want:** To show that  $B$  is recognizable. To do this we need a TM  $N$  that recognizes  $B$ , meaning for all  $w \in \Sigma^*$

- If  $0^{|w|} \in A$  or  $1^{|w|} \in A$  then  $N(w)$  accepts.
- If  $0^{|w|} \notin A$  and  $1^{|w|} \notin A$  then  $N(w)$  rejects or loops.

**Construction:** The obvious strategy is to run  $M(0^n)$  (where  $n = |w|$ ) and see if it accepts. If so,  $N(w)$  can accept. If not, it would run  $M(1^n)$  and accept if that accepts. This will not work since  $M(0^n)$  may not halt. (This might happen if  $0^n \notin A$ ). Instead, the two computations are executed in parallel. The code below details how this can be done. Namely, we run both computations for 1 step, then both for 2 steps, and so on. In this way, if either ever accepts, we will detect it and be able to accept.

$N =$  On input a string  $w$

```

 $n \leftarrow |x|$ ;  $c \leftarrow 1$ 
While ( $c \geq 1$ ) do
  Run  $M(0^n)$  for  $c$  steps
  Run  $M(1^n)$  for  $c$  steps
  If either of these computations have accepted, then accept
   $c \leftarrow c + 1$ 
EndWhile

```

**Problem 4 [20 points]** Prove that the following language is decidable:

$$A = \{ \langle D, N \rangle : D \text{ is an DFA and } N \text{ is a NFA and } L(D) \subseteq L(N) \}.$$

**Want:** To show that  $A$  is decidable, meaning to construct a TM  $M$  that decides  $A$ . This means that for all NFAs  $N$  and DFAs  $D$  we want

- If  $L(D) \subseteq L(N)$  then  $M(\langle D, N \rangle)$  accepts
- If  $L(D) \not\subseteq L(N)$  then  $M(\langle D, N \rangle)$  rejects.

**Construction:** In class and the book we showed that  $EQ_{\text{DFA}}$  is decidable. The problem is very similar, being in fact just a simpler version of this.

We note that if  $A, B$  are sets then  $A \subseteq B$  if and only if  $A \cap \overline{B} = \emptyset$ . For us the role of  $A$  is played by  $L(D)$  and the role of  $B$  is played by  $L(N)$ . The strategy is thus to first construct a DFA  $F$  such that  $L(F) = L(D) \cap \overline{L(N)}$ . Then we can test whether  $L(F) = \emptyset$  using known TMs.

Let us now give the construction. We use various TMs from the Computability Crib Sheet, which was provided to you on the quiz and is reproduced at the end of this document, as subroutines. Our TM is:  $M =$  On input  $\langle D, N \rangle$

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Let  $\langle D_1 \rangle \leftarrow T_{\text{nfa} \rightarrow \text{dfa}}(\langle N \rangle)$ 
Let  $\langle D_2 \rangle \leftarrow T_{\text{dfa}}^{\text{comp}}(\langle D_1 \rangle)$ 
Let  $\langle F \rangle \leftarrow T_{\text{dfa}}^{\cap}(\langle D, D_2 \rangle)$ 
If  $M_{\text{dfa}}^{\emptyset}(\langle F \rangle)$  accepts then accept else reject

```

**Problem 5 [20 points]** Prove that the following language is recognizable:

$$A = \{ \langle D, G \rangle : D \text{ is a DFA and } G \text{ is a CFG and } L(D) \cup L(G) \neq \{0, 1\}^* \}.$$

**Want:** To show that  $A$  is recognizable, meaning to construct a TM  $M$  that recognizes  $A$ . This means that for all DFAs  $D$  and CFGs  $G$  we want

- If  $L(D) \cup L(G) \neq \{0, 1\}^*$  then  $M(\langle D, G \rangle)$  accepts
- If  $L(D) \cup L(G) = \{0, 1\}^*$  then  $M(\langle D, G \rangle)$  rejects or loops.

**Construction:** Using various TMs from the Computability Crib Sheet as subroutines, our TM is:  
 $M =$  On input  $\langle D, G \rangle$

```

 $G_1 \leftarrow T_{\text{dfa} \rightarrow \text{cfg}}(\langle D \rangle)$ 
 $G_2 \leftarrow T_{\text{cfg}}^{\cup}(\langle G_1, G \rangle)$ 
For all  $w \in \{0, 1\}^*$  do
    If  $M_{\text{cfg}}(\langle G_2, w \rangle)$  rejects then accept
EndFor

```

We have constructed  $G_2$  so that  $L(G_2) = L(D) \cup L(G)$ . So it suffices now to determine whether or not  $L(G_2) = \{0, 1\}^*$ . For each  $w \in \{0, 1\}^*$  we test whether or not  $w \in L(G_2)$ . If we find a  $w \notin L(G_2)$  we may accept, since it means  $L(G_2) \neq \{0, 1\}^*$ . As long as we do not find such a  $w$  however, we keep looking, meaning our computation will not halt. This means the conditions stated above for  $M$  are met.

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TM	Input	Output	Property
$T_{\text{dfa}}^{\text{comp}}$	$\langle D_1 \rangle$ where $D_1$ is a DFA	$\langle D_2 \rangle$ where $D_2$ is a DFA	$L(D_2) = \overline{L(D_1)}$
$T_{\text{dfa}}^{\cap}$	$\langle D_1, D_2 \rangle$ where $D_1, D_2$ are DFAs	$\langle D \rangle$ where $D$ is a DFA	$L(D) = L(D_1) \cap L(D_2)$
$T_{\text{dfa}}^{\cup}$	$\langle D_1, D_2 \rangle$ where $D_1, D_2$ are DFAs	$\langle D \rangle$ where $D$ is a DFA	$L(D) = L(D_1) \cup L(D_2)$
$T_{\text{dfa}}^{\bullet}$	$\langle D_1, D_2 \rangle$ where $D_1, D_2$ are DFAs	$\langle D \rangle$ where $D$ is a DFA	$L(D) = L(D_1) \cdot L(D_2)$
$T_{\text{dfa}}^*$	$\langle D_1 \rangle$ where $D_1$ is a DFA	$\langle D_2 \rangle$ where $D_2$ is a DFA	$L(D_2) = L(D_1)^*$
$T_{\text{nfa} \rightarrow \text{dfa}}$	$\langle N \rangle$ where $N$ is a NFA	$\langle D \rangle$ where $D$ is a DFA	$L(D) = L(N)$
$T_{\text{cfg}}^{\cup}$	$\langle G_1, G_2 \rangle$ where $G_1, G_2$ are CFGs	$\langle G \rangle$ where $G$ is a CFG	$L(G) = L(G_1) \cup L(G_2)$
$T_{\text{cfg}}^{\bullet}$	$\langle G_1, G_2 \rangle$ where $G_1, G_2$ are CFGs	$\langle G \rangle$ where $G$ is a CFG	$L(G) = L(G_1) \cdot L(G_2)$
$T_{\text{cfg}}^*$	$\langle G_1 \rangle$ where $G_1$ is a CFG	$\langle G_2 \rangle$ where $G_2$ is a CFG	$L(G_2) = L(G_1)^*$
$T_{\text{dfa} \rightarrow \text{cfg}}$	$\langle D \rangle$ where $D$ is a DFA	$\langle G \rangle$ where $G$ is a CFG	$L(G) = L(D)$
$T_{\text{cfg} \rightarrow \text{cnf}}$	$\langle G \rangle$ where $G$ is a CFG	$\langle G' \rangle$ where $G'$ is a CFG in CNF	$L(G) = L(G')$

Figure 1: Transform-computing TMs you may use in this quiz. CNF stands for Chomsky Normal Form.

TM	Language that it decides
$M_{\text{dfa}}$	$A_{\text{DFA}} = \{ \langle D, w \rangle : D \text{ is a DFA and } D \text{ accepts } w \}$
$M_{\text{nfa}}$	$A_{\text{NFA}} = \{ \langle N, w \rangle : N \text{ is a NFA and } N \text{ accepts } w \}$
$M_{\text{dfa}}^{\emptyset}$	$E_{\text{DFA}} = \{ \langle D \rangle : D \text{ is a DFA and } L(D) = \emptyset \}$
$M_{\text{dfa}}^{\overline{=}}$	$EQ_{\text{DFA}} = \{ \langle D_1, D_2 \rangle : D_1, D_2 \text{ are DFAs and } L(D_1) = L(D_2) \}$
$M_{\text{cfg}}$	$A_{\text{CFG}} = \{ \langle G, w \rangle : G \text{ is a CFG and } w \in L(G) \}$
$M_{\text{cfg}}^{\emptyset}$	$E_{\text{CFG}} = \{ \langle G \rangle : G \text{ is a CFG and } L(G) = \emptyset \}$

Figure 2: Language-deciding TMs you may use in this quiz.