Dimension of $\mathbb{R}^\omega = 2^{\aleph_0}$

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October 4, 2003

Let $\omega = \{0, 1, \dots\}$ be the natural numbers and consider the vector space $V = \mathbb{R}^\omega$. We will show that $\dim V = 2^{\aleph_0}$. This is slightly (but not too much) counter to intuition - there is after all an ω in the exponent and we are accustomed to $\dim \mathbb{R}^n = n$.

Theorem 1. $\dim \mathbb{R}^\omega = 2^{\aleph_0}$.

Proof. To upperbound the dimension, notice that $|\mathbb{R}^\omega| = |(2^\omega)^\omega| = |2^{\omega^2}| = |2^\omega| = 2^{\aleph_0}$. So $\dim V \le |V| = 2^{\aleph_0}$.

To lowerbound the dimension, we construct an explicit set of $2^{\aleph_0}$ linearly independent vectors. We define $f : 2^\omega \to \mathbb{R}^\omega$ as follows. Let $f(x)(0) = 0$ and for $2^{k-1} \le i < 2^k$, let

$$f(x)(i) = \begin{cases} 1 & \text{if } i - 2^{k-1} = \sum_{j=0}^{k-1} 2^{x(j)} \\ 0 & \text{else} \end{cases}.$$

The idea here is that we divide ω into intervals. For $k = 1, 2, \dots$, the k^{th} interval has size 2^{k-1}. We encode $x : \omega \to 2$ into $y \in \mathbb{R}^\omega$ as follows: the k^{th} interval of y will encode the restriction of x to $k = \{0, \dots, k-1\}$ by writing $x \mid k$ out in binary as b and then putting a 1 into the b^{th} position of the k^{th} interval. All other entries of y are 0.

Let $x_1, \dots, x_n \in 2^\omega$ be distinct. Then $\exists k - 1 \in \omega$ such that $x_1 \mid k, \dots, x_n \mid k$ are distinct as well. In the k^{th} interval, $f(x_i)$ will have all 0's except for a single 1 in position $b_i = \sum_{j=0}^{k-1} 2^{x_i(j)}$. But the b_i's are distinct, and so the $f(x_i)$ are independent. $\square$