Dual of a Vector Space

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1 What is a dual space?

Let V be a vector space over a field F. Then the *dual* of V is

$$V^* = \mathrm{Hom}_F(V, F),$$

the set of F-linear homomorphisms from V to F, and indeed this is a vector space over F.

2 V, V^* isomorphic when $\dim_F V < \infty$

Claim 1. *Suppose* $\dim_F V < \infty$. *Then* $V \cong V^*$.

Proof. Let $B = \{b_1, \ldots, b_n\}$ be a basis for V. Define $\phi : V \to V^*$ by

$$\phi(\sum a_i b_i) = (b_i \mapsto a_i),$$

where the $a_i \in F$. In other words, $\phi(\sum a_i b_i)$ is the function $g : V \to F$ defined by $g(\sum c_i b_i) = \sum c_i a_i$, where the $c_i \in F$.

We claim that ϕ is an isomorphism. First let us show that ϕ is a homomorphism:

$$\begin{aligned}
\phi(\sum a_i b_i + \sum a_i' b_i) &= \phi(\sum (a_i + a_i') b_i) \\
&= (b_i \mapsto a_i + a_i') \\
&= (b_i \mapsto a_i) + (b_i \mapsto a_i') \\
&= \phi(\sum a_i b_i) + \phi(\sum a_i' b_i)
\end{aligned}$$

and

$$\begin{aligned}
\phi(c \sum a_i b_i) &= \phi(\sum c a_i b_i) \\
&= (b_i \mapsto c a_i) \\
&= c(b_i \mapsto a_i) \\
&= c\phi(\sum a_i b_i).
\end{aligned}$$

Next let us look at the kernel of ϕ. Suppose $\phi(\sum a_i b_i) = (b_i \mapsto 0)$. Then each $a_i = 0$ and hence $\sum a_i b_i = 0$. So ϕ is injective.

Next we show that ϕ is surjective. Let $g \in \mathrm{Hom}_F(V, F)$ and suppose $g(b_i) = a_i$ for each $i \in \{1, \ldots, n\}$. Then $\phi(\sum a_i b_i) = g$. Notice carefully that it is here that we use the assumption that $n < \infty$, since this allows the sum to be finite.

So indeed, ϕ is an isomorphism. □

3 V, V^* not necessarily isomorphic when $\dim_F V = \infty$

We will present a counterexample. We will use the fact that the cardinals are well ordered. In particular, let ω be the least infinite cardinal.

Recall the cardinal arithmetic theorem: if $\alpha \leq \beta$ are cardinals and $\beta \geq \omega$, then $\alpha + \beta = \alpha \cdot \beta = \beta$. Let $F = 2 = \{0, 1\}$ be the field of size 2. Then we claim that $V = 2^\omega \not\cong V^*$.

To see this, we first find the dimension of V. Let $v_i \in V$ be the vector with a 1 in the i^{th} position and 0's elsewhere. Then the v_i are linearly independent and so the dimension of V is infinite. Let B be a basis for V.

$$\begin{aligned}
|V| &= \Big| \bigcup_{n\in\omega} \bigcup_{\substack{C \subseteq B \\ |C| = n}} \Big\{ \sum_{i=1}^{n} a_i c_i \mid a_i \in 2, c_i \in C \Big\} \Big| \\
&\leq \sum_{n\in\omega} \sum_{\substack{C \subseteq B \\ |C| = n}} 2^n \\
&\leq \sum_{n\in\omega} |B|^n 2^n \\
&= \sum_{n\in\omega} |B| \quad \text{by cardinal arithmetic and } |B| \geq \omega \\
&= \omega \cdot |B| \\
&= |B| \quad \text{by cardinal arithmetic and minimality of } \omega \\
&\leq |V|.
\end{aligned}$$

So $|B| = |V| = 2^\omega$. Now $|V^*| = |\mathrm{Hom}_F(V, F)| = |F|^{\dim_F V} = 2^{2^\omega} > 2^\omega = |V|$, and so $V \not\cong V^*$.